For each $k = 1, \ldots, P$, determine num(k) = number of $c(j)$ satisfying $|c(k) - c(j)| < EP$.
Compute max_cluster = MAX {num(k): $k = 1, \ldots, P$}.
  If (max_cluster = 1)
    No match found
    Return
  End If
  If (max_cluster > M)
    More than one match-up of B with A exists (A requires thinning)
    Return
  Else
    If (more than one k satisfies num(k) = max_cluster)
      More than one optimal offset c exists
      Return
    Else
      Compute c = (average of $c(j)$, which satisfies $|c(k) - c(j)| < EP$), where k is unique and satisfies num(k) = max_cluster
      Output c (unique)
    End if
    Do $j = 1, M$
    Do $i = 1, N$
      If (i satisfies $|a(i) - b(j) - c| < EP$)
        $a(i)$ matches $b(j)$
        Output matched pair $[b(j), a(i)]$
      End if
    End Do
    End Do
  End If

Example (Phase Matching)

A visual star sensor onboard an Earth-pointing, spinning satellite is positioned at right angles to the satellite spin axis and sweeps out a 360-deg strip of sky in one sensor scan period, equal to SCAN_PERIOD seconds. Within a given scan period the sensor records $M + 1$ star sightings at times $t(1), \ldots, t(M + 1)$. The method of point stacking is used to estimate the satellite phase angle c (angle of sensor from North) at time $t(1)$, as follows. A partial catalog of star azimuth values $A = \{a(1), \ldots, a(N)\}$ associated with the approximate satellite ephemeris at $t(1)$ is available, where each azimuth lies in the range [0, 360] deg. A candidate subset (superset) of rotationally equivalent angles is given by $B = \{b(1), \ldots, b(M)\}$, where $b(i)$ = SPIN_RATE * $[t(i + 1) - t(1)]$, and SPIN_RATE = 360/SCAN_PERIOD. Sets A and B, together with an appropriate value of clustering window size EP, define the necessary algorithm inputs as outlined in the previous paragraph, where the algorithm itself has been modified to account for 360-deg angular wraparound in all instances where angles are to be compared for EP closeness. The algorithm output c defines the desired estimate of satellite phase angle at $t(1)$.

A specific application is afforded by selecting SCAN_PERIOD = 10 s $M = 7$, $EP = 0.01$ deg, sighting times of 1.0000, 1.0416, 1.8194, 6.0972, 7.0138, 10.4861, and 10.8194 s, and $N = 8$ catalog azimuths, given by 5, 350, 1, 345, 9, 33, 180, 220 deg. (Normally the azimuth values will be sorted, but point stacking does not require this.) The six-element B set is computed as 1.4976, 29.4984, 183.4992, 216.4968, 341.4996, and 353.4984 deg. Point stacking returns the phase angle estimate $c = 3.5019$ deg at time $t(1) = 1.0000$ s and shows that the fourth and seventh star sightings at times 6.0972 and 10.8194 s are not optimally matchable with members of the given catalog set and are probably spurious.

Conclusions

A dependable algorithm called point stacking has been developed for use as a general purpose software tool for matching two input sets of vectors. The algorithm is not sensitive to false data in either set and does not require that the sets be sorted. In effect, the algorithm estimates the most likely matchup of vectors by computing the largest possible subsets that are translationally equivalent. The method finds immediate application to satellite phase matching and Earth point-source recognition problems.

Acknowledgments

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Orbital Motion Under Continuous Tangential Thrust

Frederick W. Boltz*
NASA Ames Research Center, Moffett Field, California 94035

Introduction

THERE are relatively few problems of continuous thrusting by spacecraft in orbit for which an exact (or even a nearly exact) analytic solution for the orbital motion and mass loss is possible.[1,2] One such problem is that of optimal escape from circular orbit, which has received a good deal of attention in past years[3–6] (with notable but limited success in describing the precise motion at moderate and low levels of thrust). The pioneering work of Tsien,[3] who considered circumferential thrust in finding an approximate solution to this problem, was followed by that of Benney,[4] who considered tangential thrust, and that of Lawden,[5,6] who determined the optimum direction of thrust for minimizing expenditure of rocket propellant. It was found that, for all practical purposes, optimum thrust is tangential (or in the flight-path direction) as assumed by Benney. However, as in the solution found by Tsien, the detailed solutions for the flight path and mass loss obtained by Benney and Lawden apply only to cases of large or small thrust. Moreover, in the case of small or microthrust, the solutions obtained are valid only in the initial and intermediate portions of the escape trajectory. As noted by Lawden,[6] the assumptions required to obtain such solutions are invalid in the final portion of the trajectory as escape speed is approached and the instantaneous or osculating ellipse can no longer be considered close to a circle.

The purpose of this Note is to show how the limitations inherent in the solutions presented by Tsien, Benney, and Lawden can be eliminated if a simple change is made in the arbitrary specification of the variation of thrust magnitude. Rather than considering a constant value of tangential thrust acceleration (which is only a mathematical convenience without any practical advantage), it is assumed that the ratio of this acceleration to that of gravity is fixed. This provides a constant value of specific thrust acceleration for which the ratio of thrust to vehicle weight in orbit is fixed. The solutions

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*Aerospace Engineer (retired). Member AIAA.

obtained with this constraint are valid for any constant value of this ratio and describe motion along the full extent of the escape trajectory. At large values of the thrust-to-weight ratio, for which escape velocity is attained soon after leaving the initial orbit, the present solution gives essentially the same results as those found by Benney and Lawden. At small values of this ratio, the results are also very similar for time intervals such that the spiral flight path does not extend far beyond the initial orbit. However, when this does occur, the solution results depart significantly. Since the thrust-to-weight ratio remains fixed in the one case and increases continually in the other, there is a difference in the unwinding spiral flight paths to either reach escape speed or continue along a tight spiral path at nearly circular orbit speed. In contradistinction to the case of constant tangential thrust acceleration, there is found to be a critical value of constant thrust-to-weight ratio below which escape speed cannot be attained.

Equations of Motion

When thrust of magnitude T is directed at angle τ to the flight path of a vehicle in orbit, the equations expressing the balance of forces in and normal to the direction of motion, respectively, are

$$\mathrm{d}V/\mathrm{d}t = (T/m)\cos\tau - g\sin\gamma \tag{1}$$

$$V(\mathrm{d}\gamma/\mathrm{d}t) = (T/m)\sin\tau + (V^2/r - g)\cos\gamma \tag{2}$$

where V is the vehicle trajectory, m is its mass, t is time, γ is the flight-path angle, r is radial distance from the planet's center, and g is the acceleration of gravity. These equations may be expressed more conveniently in dimensionless form as

$$\sqrt{r/g}\,(\mathrm{d}\bar{V}/\mathrm{d}t) = (T/mg)\cos\tau - (1 - \bar{V}^2/2)\sin\gamma \tag{3}$$

$$\sqrt{r/g}\,\bar{V}(\mathrm{d}\gamma/\mathrm{d}t) = (T/mg)\sin\tau - (1 - \bar{V}^2)\cos\gamma \tag{4}$$

where $\bar{V}$ is the vehicle velocity divided by the circular orbital velocity u_c, as given by

$$\bar{V} \equiv V/u_c = V/\sqrt{gr} = V/\sqrt{\mu/r} \tag{5}$$

with μ being the gravitational parameter.

In the case of tangential thrust (in the flight-path direction), $\tau = 0$ so that Eqs. (3) and (4) reduce to

$$\sqrt{r/g}\,(\mathrm{d}\bar{V}/\mathrm{d}t) = T/mg - (1 - \bar{V}^2/2)\sin\gamma \tag{6}$$

$$\sqrt{r/g}\,\bar{V}(\mathrm{d}\gamma/\mathrm{d}t) = (\bar{V}^2 - 1)\cos\gamma \tag{7}$$

During escape from circular orbit using continuous tangential thrust, the trajectory is an unwinding spiral with the flight-path angle continuously increasing as escape speed is approached. The eccentricity of the instantaneous or osculating ellipse along this trajectory is given by

$$e = [1 - \bar{V}^2(2 - \bar{V}^2)\cos^2\gamma]^{1/2} \tag{8}$$

and, for sufficiently high levels of thrust, increases from 0 to 1 at escape speed ($\bar{V} = \sqrt{2}$).

Characteristic Velocity

If c is the effective exhaust velocity of the vehicle's rocket propulsion system, the thrust impulse is given by

$$T\,\mathrm{d}t = -c\,\mathrm{d}m \tag{9}$$

so that

$$(T/m)\,\mathrm{d}t = -c\,(\mathrm{d}m/m) \tag{10}$$

where T/m is the fixed or variable thrust acceleration. The characteristic velocity W is a measure of the propellant mass loss during a thrusting maneuver as given by

$$W = \int (T/m)\,\mathrm{d}t = c\,\ln(m_1/m_2) \tag{11}$$

It is generally expressed in normalized (dimensionless) form as

$$\bar{W} = W/\sqrt{g_1 r_1} = (c/\sqrt{g_1 r_1})\ln(m_1/m_2) \tag{12}$$

where $\sqrt{g_1 r_1}$ is the velocity along the circular orbit. Since T/mg is considered fixed in the present analysis, the value of $\bar{W}$ is obtained from

$$\bar{W} = (T/mg)(g_{\mathrm{ave}}/\sqrt{g_1 r_1})\Delta t = 4(T/mg)\sqrt{\mu/r_1^3}\,\Delta t/(1 - r_2/r_1)^2 \tag{13}$$

In the idealized case of impulsive thrust (for which the duration of thrusting is 0), the theoretical value of $\bar{W}$ is found by replacing $(T/mg)\,\mathrm{d}t$ with $\sqrt{r/g}\,\mathrm{d}\bar{V}$ [according to Eq. (6)] and using Eqs. (11) and (12) to obtain

$$\bar{W} = (g_1/\sqrt{g_1 r_1})\sqrt{r_1/g_1}\int_1^{\sqrt{2}} \mathrm{d}\bar{V} = \sqrt{2} - 1 \tag{14}$$

Approximate Analytic Solution

It is possible to find an approximate but accurate analytic solution for escape from circular orbit using continuous tangential thrust. Dividing Eq. (6) by Eq. (7) gives

$$(\bar{V} - 1/\bar{V})\,\mathrm{d}\bar{V} = [(T/mg)/\cos\gamma - (1 - \bar{V}^2/2)\tan\gamma]\,\mathrm{d}\gamma \tag{15}$$

which may be solved approximately for γ as a function of $\bar{V}$ (with T/mg considered fixed) in several ways as follows.

Higher Thrust Levels

If T/mg is fixed and greater than about ½, so that the flight-path angle does not get very large ($\gamma < 20$ deg, say), the second term on the right-hand side of Eq. (15) may be deleted with little loss of accuracy (since it is 0 initially with $\gamma = 0$ and returns to 0 as $\bar{V}$ approaches escape value $\sqrt{2}$). Thus, for the approximate variation of γ with $\bar{V}$,

$$(\bar{V} - 1/\bar{V})\,\mathrm{d}\bar{V} = [(T/mg)/\cos\gamma]\,\mathrm{d}\gamma \tag{16}$$

which may be integrated to obtain, with $\bar{V}_1 = 1$ and $\gamma_1 = 0$,

$$\tfrac{1}{2}(\bar{V}_2^2 - 1) - \ln\bar{V}_2 = (T/mg)\ln(\sec\gamma_2 + \tan\gamma_2) \cong (T/mg)\sin\gamma_2 \tag{17}$$

so that, when $\bar{V}_2 = \sqrt{2}$,

$$\sin\gamma_2 \cong (\tfrac{1}{2} - \ln\sqrt{2})/(T/mg) \tag{18}$$

Such escape values of γ_2 are presented in Table 1 for moderate-to-large values of T/mg.

The approximate change in radial distance along the escape trajectory may be found using $\mathrm{d}r/\mathrm{d}t = \sqrt{gr}\,\bar{V}\sin\gamma$ in conjunction with $\sqrt{r/g}\,(\mathrm{d}\bar{V}/\mathrm{d}t) = T/mg$ and Eq. (17) to obtain

$$(T/mg)^2(\mathrm{d}r/r) = [\tfrac{1}{2}(\bar{V}^3 - \bar{V}) - \bar{V}\ln\bar{V}]\,\mathrm{d}\bar{V} \tag{19}$$

which integrates from $\bar{V}_1 = 1$ and $\gamma_1 = 0$ into

$$(T/mg)^2\ln(r_2/r_1) = \tfrac{1}{8}(\bar{V}_2^4 - 1) - \tfrac{1}{4}(\bar{V}_2^2 - 1) - \left[\frac{\bar{V}_2^2}{2}(\ln\bar{V}_2 - \tfrac{1}{2}) + \tfrac{1}{4}\right] \tag{20}$$

so that, when $\bar{V}_2 = \sqrt{2}$,

$$r_2/r_1 = \exp\ [(3/8 - \ln\sqrt{2})/(T/mg)^2] \tag{21}$$

Such escape values of r_2/r_1 are also presented in Table 1.

The approximate time of flight along the escape trajectory may be obtained from Eq. (6) expressed as

$$\sqrt{g/r}\ \mathrm{d}t = [T/mg - (1 - \bar{V}^2/2)\sin \tfrac{1}{2}(\gamma_1 + \gamma_2)]^{-1}\,\mathrm{d}\bar{V} \tag{22}$$

which, using average values for the time parameter $\sqrt{g/r} = \sqrt{\mu/r^3}$, integrates into

$$\sqrt{\mu/r_1^3}\ \Delta t = \frac{(1 + r_2/r_1)^{3/2}}{\sqrt{2}S\ \sin(\gamma_2/2)}\,[\tan^{-1}(\bar{V}_2/S) - \tan^{-1}(\bar{V}_1/S)] \tag{23}$$

where $S = [2(T/mg)/\sin(\gamma_2/2) - 2]^{1/2}$ and the reciprocal time parameter $\sqrt{\mu/r_1^3}$ has a value of about $1/850\ \mathrm{s}^{-1}$ for low-Earth orbit. The dimensionless values of $\sqrt{\mu/r_1^3}\ \Delta t$ listed in Table 1 were obtained by using the corresponding values of γ_2 and r_2/r_1 tabulated along with $\bar{V}_1 = 1$ and $\bar{V}_2 = \sqrt{2}$ in Eq. (23).

The approximate angular rotation about the planet in radians during the thrusting period to reach escape speed is given by

$$\Delta\theta = V_{\mathrm{ave}}(\Delta t/r)\cos\gamma_{\mathrm{ave}} = \tfrac{1}{2}(\bar{V}_1 + \bar{V}_2)\sqrt{g/r}\ \Delta t\ \cos \tfrac{1}{2}(\gamma_1 + \gamma_2) \tag{24}$$

so that, with average values for the time parameter and the usual initial and final conditions,

$$\Delta\theta = \tfrac{1}{2}(1 + \sqrt{2})\sqrt{\mu/r_1^3}\ \Delta t\ \cos(\gamma_2/2)/[\tfrac{1}{2}(1 + r_2/r_1)]^{3/2} \tag{25}$$

Table 1 Flight conditions for escape from circular orbit using continuous tangential thrust and fixed thrust-to-weight ratio $\bar{V}_2 = \sqrt{2}$

T/mg	γ_2, deg	r_2/r_1	$\sqrt{\mu/r_1^3}\ \Delta t$	$\theta_2 - \theta_1$, deg	$\bar{W}$
0.2	50.097	2.03533	—	—	—
0.3	30.759	1.37142	2.39280	123.59	0.51059
0.4	22.555	1.19443	1.37577	81.19	0.45711
0.5	17.870	1.12042	0.98760	61.81	0.43931
0.6	14.816	1.08216	0.77832	50.25	0.43087
0.7	12.661	1.05973	0.64566	42.47	0.42613
0.8	11.057	1.04542	0.55329	36.83	0.42319
0.9	9.815	1.03572	0.48490	32.54	0.42123
1.0	8.826	1.02883	0.43204	29.16	0.41985
1.5	5.871	1.01271	0.28132	19.25	0.41667
2.0	4.400	1.00713	0.20928	14.39	0.41559
3.0	2.932	1.00316	0.13871	9.57	0.41482
Impulsive thrust	0	1.00000	0	0	0.41421

Such escape values of $\Delta\theta$ converted to degrees are also presented in Table 1 along with values of $\bar{W}$ obtained from Eqs. (13) and (14).

Stepwise Solution

In general, more accurate results at moderate thrust levels $(0.1 < T/mg < 0.5)$ may be obtained by using a stepwise solution to determine the flight conditions for escape from circular orbit. In this case the second term on the right-hand side of Eq. (15) is retained and $\bar{V}$ there is approximated by $\tfrac{1}{2}(\bar{V}_1 + \bar{V}_2)$ so that

$$(\bar{V} - 1/\bar{V})\,\mathrm{d}\bar{V} = \{(T/mg)/\cos\gamma - [1 - (\bar{V}_1 + \bar{V}_2)\,2/8]\tan\gamma\}\,\mathrm{d}r \tag{26}$$

which integrates into

$$\tfrac{1}{2}(\bar{V}_2^2 - \bar{V}_1^2) - \ln(\bar{V}_2/\bar{V}_1) = (T/mg)\ln\left(\frac{1 + \sin\gamma_2}{1 + \sin\gamma_1}\,\frac{\cos\gamma_1}{\cos\gamma_1}\right) + [1 - (\bar{V}_1 + \bar{V}_2)^{2/8}]\ln(\cos\gamma_2/\cos\gamma_1) = \ln\left[\left(\frac{1 + \sin\gamma_2}{1 + \sin\gamma_1}\right)^{T/mg}\left(\frac{\cos\gamma_2}{\cos\gamma_1}\right)^{1 - T/mg - (\bar{V}_1 + \bar{V}_2)2/8}\right] \tag{27}$$

with $\bar{V}_2 - \bar{V}_1 \leq 0.05$ for better accuracy. This equation can be solved by trial and error for γ_2 when $\bar{V}_1$, $\bar{V}_2$, and γ_1 are specified. Thus, the solution proceeds in stepwise fashion from circular orbital values to escape conditions. The results of several such solutions are presented in Table 2. As would be expected, comparison of γ_2 values at escape speed with those listed in Table 1 shows closer agreement with increasing value of T/mg. However, for $T/mg = 0.1$, a different kind of result is obtained. It is found that at a certain point during the stepwise solution only very small increments in $\bar{V}$ will compute to provide continued increase in γ. Moreover, it appears that a maximum value of $\bar{V}$ is reached beyond which the value of γ continues to increase with a reduction in $\bar{V}$. An explanation of this seemingly peculiar behavior is to be found in the following analysis of low-thrust flight characteristics.

Lower Thrust Levels

At lower thrust levels $(T/mg \leq 0.1)$, the flight path is roughly a logarithmic spiral, as is subsequently shown, and escape speed is never attained. In this case the flight-path angle remains small so that, with $\bar{V}_1 = 1$ and $\gamma_1 = 0$, Eq. (27) can be approximated by

$$\tfrac{1}{2}(\bar{V}_2^2 - 1) - \ln\bar{V}_2 = (T/mg)\sin\gamma_2 + [1 - (1 + \bar{V}_2)2/8] \times (\cos\gamma_2 - 1) = (T/mg)\sin\gamma_2 - \tfrac{1}{2}[1 - (1 + \bar{V}_2)2/8]\sin^2\gamma_2 \tag{28}$$

since, for small angles, $\cos\gamma = (1 - \sin^2\gamma)^{1/2} = 1 - \tfrac{1}{2}\sin^2\gamma$.

Table 2 Results of stepwise solution for escape from circular orbit using continuous tangential thrust and fixed thrust-to-weight ratio

$T/mg = 0.1$		$T/mg = 0.2$		$T/mg = 0.3$		$T/mg = 0.4$		$T/mg = 0.5$	
$\bar{V}_2$	γ_2, deg	$\bar{V}_2$	γ_2, deg	$\bar{V}_2$	γ_2, deg	$\bar{V}_2$	γ_2, deg	$\bar{V}_2$	γ_2, deg
1.050	1.503	1.05	0.715	1.05	0.473	1.05	0.354	1.05	0.283
1.100	7.769	1.10	2.934	1.10	1.895	1.10	1.406	1.10	1.120
1.110	10.817	1.15	6.933	1.15	4.304	1.15	3.159	1.15	2.503
1.111	11.261	1.20	13.191	1.20	7.752	1.20	5.617	1.20	4.429
1.112	11.763	1.25	22.166	1.25	12.243	1.25	8.765	1.25	6.877
1.113	12.350	1.30	33.159	1.30	17.635	1.30	12.545	1.30	9.817
1.114	13.088	1.35	43.452	1.35	23.581	1.35	16.826	1.35	13.182
1.115	14.242	1.40	51.298	1.40	29.599	1.40	21.412	1.40	16.870
1.116	17.507	1.414	53.107	1.414	31.257	1.414	22.738	1.414	17.959

Table 3 Results of low-thrust solution for the effects of continuous tangential thrust applied to a vehicle in circular orbit

$T/mg = 0.001$			$T/mg = 0.002$			$T/mg = 0.005$		
$\bar{V}_2$	γ_-, deg	γ_+, deg	$\bar{V}_2$	γ_-, deg	γ_+, deg	$\bar{V}_2$	γ_-, deg	γ_+, deg
0.9991	0.065	0.164	0.9981	0.158	0.299	0.9955	0.325	0.816
0.9992	0.046	0.183	0.9982	0.130	0.328	0.9960	0.230	0.911
0.9994	0.023	0.206	0.9985	0.078	0.380	0.9970	0.115	1.028
0.9996	0.010	0.220	0.9990	0.031	0.427	0.9980	0.048	1.096
0.9998	0.002	0.227	0.9995	0.007	0.451	0.9990	0.012	1.133
1.0000	0.000	0.229	1.0000	0.000	0.458	1.0000	0.000	1.146
1.0002	0.002	0.227	1.0005	0.007	0.451	1.0010	0.012	1.136
1.0004	0.010	0.220	1.0010	0.031	0.428	1.0020	0.048	1.100
1.0006	0.023	0.206	1.0015	0.078	0.382	1.0030	0.114	1.035
1.0008	0.046	0.184	1.0018	0.129	0.330	1.0040	0.228	0.922
1.0009	0.065	0.165	1.0019	0.157	0.302	1.0045	0.321	0.830
1.0010	0.111	0.119	1.0020	0.218	0.241	1.0050	0.529	0.623
$0 \le e \le 0.004$			$0 \le e \le 0.008$			$0 \le e \le 0.020$		
$T/mg = 0.01$			$T/mg = 0.02$			$T/mg = 0.05$		
$\bar{V}_2$	γ_-, deg	γ_+, deg	$\bar{V}_2$	γ_-, deg	γ_+, deg	$\bar{V}_2$	γ_-, deg	γ_+, deg
0.991	0.654	1.618	0.981	1.635	2.865	0.955	3.429	7.566
0.992	0.461	1.813	0.982	1.322	3.183	0.960	2.368	8.689
0.994	0.230	2.049	0.985	0.784	3.735	0.970	1.164	10.015
0.996	0.096	2.187	0.990	0.308	4.234	0.980	0.482	10.818
0.998	0.023	2.265	0.995	0.073	4.493	0.990	0.116	11.303
1.000	0.000	2.292	1.000	0.000	4.589	1.000	0.000	11.537
1.002	0.023	2.274	1.005	0.073	4.539	1.010	0.115	11.538
1.004	0.096	2.206	1.010	0.306	4.328	1.020	0.475	11.293
1.006	0.228	2.078	1.015	0.768	3.889	1.030	1.129	10.750
1.008	0.456	1.855	1.017	1.068	3.598	1.040	2.224	9.765
1.009	0.640	1.673	1.019	1.526	3.148	1.050	4.491	7.606
1.010	1.024	1.291	1.020	1.958	2.721	1.051	5.010	7.097
$0 \le e \le 0.040$			$0 \le e \le 0.080$			$0 \le e \le 0.200$		

Then, using the quadratic formula,

$$\sin \gamma_2 = \frac{T/mg \pm \{(T/mg)^2 - (\bar{V}_2^2 - 1 - 2 \ln \bar{V}_2)[1 - (1 + \bar{V}_2)^2/8]\}^{1/2}}{1 - (1 + \bar{V}_2)^2/8} \tag{29}$$

where the $\pm$ sign indicates that $\sin \gamma_2$ is a double-valued function of $\bar{V}_2$ for a small range of values of $\bar{V}_2$ above and below 1. This suggests that there is a maximum and a minimum value of $\bar{V}_2$, which is, in fact, the case as is clear from the results presented in Table 3. Here are listed, for several fixed values of T/mg, the two values of γ_2 obtained from Eq. (29) for a number of acceptable values of $\bar{V}_2$. The proper sequential order for each set of results is down the column labeled γ_- and up the column labeled γ_+. Thus, with T/mg fixed at a small value, the flight path is that where the flight-path angle repeatedly increases from 0 to a maximum value twice that for a logarithmic spiral trajectory (see Appendix) and then decreases back to 0. This cycle is continually repeated with precisely the same values of $\bar{V}$ and γ, since a point is always reached at a higher altitude where the initial values ($\bar{V}_1 = 1$ and $\gamma_1 = 0$) are again attained. The value of eccentricity for the osculating ellipse along the flight path, as given by Eq. (8), increases from 0 to that given by the sine of the maximum flight-path angle. The angular rotation about the planet during one complete cycle or excursion in γ from 0 is found using Eq. (24) to be about 360 deg, since $\frac{1}{2}(\bar{V}_1 + \bar{V}_2) = 1$, $\cos \frac{1}{2}(\gamma_1 + \gamma_2) = \cos(\gamma_{max}/2)$, and $\sqrt{g/r}\,\Delta t = \sqrt{\mu/r^3}\,\Delta t = 2\pi$ (according to Kepler's law).

Critical Value of T/mg

A critical value of T/mg, below which escape speed is not attained, may be found by considering Eq. (29). When a maximum or a minimum value of $\bar{V}_2$ occurs at low thrust level, the value of $\sin \gamma_2$ is single valued, which means the radical on the right-hand side of Eq. (29) disappears. Thus, for this condition,

$$(T/mg)^2 = (\bar{V}_2^2 - 1 - 2 \ln \bar{V}_2)(1 - \bar{V}_2^2/2) \tag{30}$$

Now, assuming that there is a maximum value of T/mg for this condition to exist, it is found by first setting the derivative of T/mg with respect to $\bar{V}_2$ equal to 0 so that

$$(\bar{V}_2 - 1/\bar{V}_2)(2 - \bar{V}_2^2) = \bar{V}_2^3 - \bar{V}_2 - 2\bar{V}_2 \ln \bar{V}_2 \tag{31}$$

which may be solved by trial and error to obtain $\bar{V}_2 = 1.277692$. Then, with this value of $\bar{V}_2$, Eq. (30) gives 0.16175 for the critical value of T/mg.

Conclusions

An approximate analytic solution has been obtained for the effects of continuous tangential thrust (in the flight-path direction) on the orbital motion and mass loss of a vehicle initially in a circular orbit. It is found that, with continuous application of tangential thrust equal in magnitude to a fixed fraction of the vehicle weight (as the product of vehicle mass and the ambient acceleration of gravity), the flight path can take one of two possible forms. If the value of thrust-to-weight ratio is greater than 0.16175, escape speed will eventually be reached along an unwinding spiral trajectory. If the value is less than 0.16175, escape speed is never attained, and the flight path oscillates around a logarithmic spiral trajectory. Formulas have been found for the approximate orbital motion and time of flight along each type of trajectory and for mass loss due to expenditure of rocket propellant (based on the effective exhaust velocity of the propulsion system).

Appendix: Logarithmic Spiral Trajectory

A logarithmic spiral flight path is that for which the flight-path angle is constant. When this condition prevails (using tangential thrust with T/mg fixed), it is readily seen from Eqs.

(6–8) that $\bar{V} = 1$, $T/mg = ½ \sin \gamma$, and $e = \sin \gamma$. Therefore, it follows that the velocity decreases along the spiral path but is always equal to the value of local circular orbital speed whereas the value of T/mg must remain fixed along with the eccentricity of the osculating ellipse. From the results presented in Table 3 for the effects of continuous low thrust (with T/mg fixed) applied to a vehicle in circular orbit, it is found that the maximum value of flight-path angle is almost exactly twice the value for the corresponding logarithmic spiral so that the average value of flight-path angle is about the same as the fixed spiral angle. Since the average value of $\bar{V}$ in each case presented in Table 3 is also the same as the fixed value for the logarithmic spiral, it is apparent that the flight path in each case of continuous low thrust may be considered to oscillate around the corresponding logarithmic spiral trajectory.

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Optimum Rendezvous Transfer Between Coplanar Heliocentric Elliptic Orbits Using Solar Sail

P. V. Subba Rao* and R. V. Ramanan†

Vikram Sarabhai Space Centre,

Trivandrum 695022, India

Introduction

IT is well recognized that solar-sail vehicles, using light pressure for propulsion, offer an attractive mode of travel for interplanetary space probes. The parameter that best indicates sail spacecraft's performance capability is the characteristic acceleration α, which the spacecraft would experience at a unit distance (1 AU) from the sun when the sail is oriented normal to sunlight. Feasible values for α seem to be 0.5–3.0 mm/s^2. For an idealized, perfectly reflecting flat sail whose area and spacecraft mass remain invariant with time, the local acceleration of the spacecraft would vary by the square of the cosine of the sail orientation angle (θ angle between the sail's outward drawn normal and the radial direction) and inversely as the square of the heliocentric distance, r. It is interesting to evaluate the time history of θ that would fulfill a certain mission in a prescribed optimal manner. The present study is concerned with the time-optimal rendezvous transfer trajectories. The analysis, however, is restricted to a two-body inverse square law of force field model in two dimensions. Employing calculus of variations in the form of "maximum principle," Zhukov and Lebedev[1] presented a minimum time strategy for transfer between two coplanar, circular, heliocentric orbits. The present attempt is an expansion to include elliptic orbits for the terminals. The formulation is attempted through a set of autonomous variables L, Φ, and Ψ [see Eq. (3)] related to angular momentum h and the Cartesian components of the eccentricity vector. As in Zhukov and Lebedev, the optimal strategy is pursued by conversion to a two-point boundary-value problem for a system of seven ordinary differential equations. Its solution is attempted with the controlled random search (CRS) optimization technique. It can be noted that Sauer[2] has presented a formulation in terms of the position and velocity vectors in three dimensions. However, the present formulation brings out explicitly the effect of the eccentricities of the terminal orbits on the time-optimal transfer.

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*Senior Scientist, Applied Mathematics Division.

†Scientist, Applied Mathematics Division.

Model for Space Vehicle Motion

Following the approach of vector techniques,[3] we consider the equations of motion of a spacecraft with an idealized, perfectly reflecting flat sail, in terms of the radial and transversal components Q and R, respectively, of the perturbing acceleration:

$$h' = h^5 r/P^3 \tag{1a}$$

$$\delta' = \left[PQ \sin\phi + R(1+P)\cos\phi + R\,\delta\right] h^4/P^3 \tag{1b}$$

$$\epsilon' = \left[-PQ \cos\phi + R(1+P)\sin\phi + R\epsilon\right] h^4/P^3 \tag{1c}$$

where

$$P = 1 + \delta\cos\phi + \epsilon\sin\phi, \qquad r = h^2/P$$

$$Q = \alpha\cos^3\theta/r^2, \qquad R = \alpha\cos^2\theta\sin\theta/r^2$$

and δ, ϵ are the components $e\cos\omega$, $e\sin\omega$ of the eccentricity (e) vector, ω being the argument of perihelian. The prime indicates differentiation with respect to the angular position variable ϕ of the vehicle; this is related to the time t by

$$t' = h^3/P^2 \tag{2}$$

The units employed for the distance and velocity are astronomical units (1496×10^5 km) and Earth-mean-orbital speed (29.78 km/s). Although this description provides a good stable foundation for orbit development, the equations are considerably simplified[4] when we transform the parameters h, δ, and ϵ to the variables defined by

$$L = \ell n(h^2) \tag{3a}$$

$$\Phi = \delta\cos\phi + \epsilon\sin\phi = e\cos\nu \tag{3b}$$

$$\Psi = \delta\sin\phi - \epsilon\cos\phi = e\sin\nu \tag{3c}$$

where ν is the true anomaly. The transformed equations of motion are

$$L' = 2\alpha\cos^2\theta\sin\theta/P \tag{4a}$$

$$\Phi' = 2\alpha\cos^2\theta\sin\theta - \Psi \tag{4b}$$

$$\Psi' = \Phi + \alpha\cos^2\theta[\cos\theta + \sin\theta\Psi/P] \tag{4c}$$

$$t' = \exp(3L/2)/P^2 \tag{4d}$$

Problem Formulation

Initially (at $t = 0$), the spacecraft is in the same heliocentric elliptic orbit as that of the Earth (departure planet), with the initial conditions

$$\phi = \phi_1, \qquad L = L_1 = \ell n(h_1^2) \tag{5a}$$